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The Lyapunov exponents in a periodic window for a weak-coupled map lattice

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Abstract. The first Lyapunov exponent in a period window for a weak-coupled map lattice is calculated. Within the windows the behaviour of the coupled map lattice could be recovered by considering a small number of modes. The depth of the windows is well defined.

1. Introduction

It is well known [1–4] that both periodic and chaotic motion could take place in a deterministic dynamical system such as a nonlinear map or a set of nonlinear differential equations. Recently much attention has turned to the so-called temporal-spatial chaos in a coupled map lattice and in nonlinear partial differential equations. In this paper we discuss a standard one-dimensional diffusively coupled map lattice with the dynamics given by [5–7]

$$x_{n+1}^{(i)} = (1-\epsilon)f(x_n^{(i)}) + \frac{\epsilon}{2}[f(x_n^{(i-1)}) + f(x_n^{(i+1)})] . \tag{1}$$

here $x_n^{(i)} \in [0,1]$ is the state at the lattice point i $(i = 1, 2, \ldots, L)$ at a discrete time step n, and L the system size. We always assume periodic boundary conditions, i.e.

$$x_n^{(0)} = x_n^{(L)} \qquad \forall n . \tag{2}$$

The parameter ϵ in (1) is called the coupling strength. In the case of $\epsilon = 0$ the map lattice (1) is called an uncoupled map lattice, and all point in the lattice then becomes independent of one other. The local mapping function $f(x)$, for instance, can be chosen as a unimodal map [5–7]

$$f(x) = \mu x(1-x) . \tag{3}$$

The dynamics of this map could be either periodic or chaotic, depending on the nonlinearity μ. The region $1 \leqslant \mu < \mu_\infty \equiv 3.5699456\ldots$ may be called the

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bifurcation region, where the dynamics is always periodic, and a bifurcation cascade is found. On the other hand, for $\mu_\infty \leqslant \mu \leqslant 4$ the iteration does not have to be periodic, and this region is called the chaotic region. With $\mu = 4.0$ the function maps the interval [0, 1] exactly onto itself, the map is then said to be 'complete', and the dynamics is in a fully developed chaotic state. However, for some values of μ in the chaotic region a periodic motion is still possible, and such intervals of μ are called periodic windows.

Chaotic or periodic motion in a single map is characterized by a positive or negative Lyapunov exponent defined by

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \ln |f'(x_n)| \tag{4}$$

with x_0 the starting point. For the coupled maps (1) a set of Lyapunov exponents can similarly be defined by considering the variation

$$\mathrm{d}x_{n+1}^{(i)} = \sum_{j=1}^{L} A_{ij} \mathrm{d}x_n^{(j)} \tag{5}$$

where

$$A_{ij} = (1-\epsilon) f'(x_n^{(i)}) \delta_{ij} + \frac{\epsilon}{2} [f'(x_n^{(i-1)}) \delta_{i-1\,j} + f'(x_n^{(i+1)}) \delta_{i+1\,j}] \, . \tag{6}$$

The system (1) has L Lyapunov exponents. The largest one, often called the first Lyapunov exponent, describes the growth of distance analogously to the single-map case. Numerically this exponent can be calculated by the standard technique [8].

In a recent work [9] the authors found that for the complete maps there is a universal relation between the first Lyapunov exponent Λ and the coupling strength ϵ:

$$\Lambda \simeq \lambda_0 + \kappa \epsilon^\sigma \qquad \text{for} \quad \epsilon \to 0 \, . \tag{7}$$

Here λ_0 and κ depend on the details of the maps, and σ is a universal constant

$$\sigma = 1/p \tag{8}$$

the with p the order of the local map's maximum. This scaling law, however, is no longer valid for the value of the parameter μ in a period window. For instance, at $\mu = \mu_0 \equiv 1 + \sqrt{8} = 3.8284\ldots$ a tangent bifurcation occurs. With a larger value of μ the stable orbit of the single map has period 3. At $\mu = \mu_b \equiv 3.8415\ldots$ a fork bifurcation takes place, and the period of the stable orbit becomes 6. Hence, as

$$\mu_0 < \mu < \mu_b \tag{9}$$

the stable state of iteration (3) is a period-3 orbit. In this paper we will mainly consider this period-3 window. If the parameter μ takes a value in the interval (9) the spatially homogeneous and temporally periodic state is linearly stable, as can be easily checked. This window should be observed for initial conditions in the vicinity of the homogeneous state. For random initial values in a coupled map lattice of large size, however, as long as the coupling strength is not very weak, it takes an extremely long time (increasing more rapidly than e^L) to settle down to the stable coherent state [7, 10]. The stable homogeneous period-3 motion then could hardly be observed in a numerical experiment. The window is distroyed for almost all initial conditions. Mechanisms of this destruction are spatiotemporal intermittency and supertransients [7]. The value obtained in a numerical simulation for the first Lyapunov exponent is often positive, and is called 'finite-time Lyapunov exponent'. In the case of very weak coupling strength ϵ (about 10^{-3}) varied stable period-3 motion might be found, which could be either homogeneous or not, after a short transient time ($\propto \ln L$). It implies that the behaviour of the map lattice system with very weak coupling strength is quite complicated. Because of this complicated nature the detailed structure of the period window has never been studied. In this paper we will discuss the dependence of the first Lyapunov exponent on the coupling strength in the period windows.

In section 2 the first Lyapunov exponent Λ is calculated by the standard technique, and the dependence of the values Λ on the strength of coupling ϵ are shown to be zigzag-like. In section 3 a mode-analytic approach for calculating the first Lyapunov exponent is proposed, and the zigzag Λ–ϵ curve is explained by the new approach. Some discussions are included in the last section. In the present paper we will only discuss the coupled map lattice (1) with local dynamics (3). Results presented here, however, could be generalized to other maps and other couplings.

2. Numerical simulation

Let us at first observe a numerical result. Taking $L = 173$ and $\mu = 3.831$, keeping a value of ϵ within the interval [0, 0.002], we calculate the first Lyapunov exponent Λ over 1440 iterations after 8000 transient iterations. The initial values for the lattice are vested randomly. The results are shown in figure 1, where the Λ–ϵ curve is zigzag-like in the interval $[0, \epsilon_w]$ with $\epsilon_w \sim 1.22 \times 10^{-3}$. The negative first Lyapunov exponent in this interval means that the motion of the map lattice system is periodic. When $\epsilon > \epsilon_w$ we find that $\Lambda > 0$, and the motion becomes chaotic. The fact that the variation of Λ in order of 10^{-1} is caused by a small variation of ϵ in order of 10^{-4} implies that the very weak coupling could change the behaviour of the system distinctly.

The elements A_{ij} in the matrix (6) alter their values as increasing the coupling strength. In one aspect the values of $f(x_n^{(i)})$ might change since the orbits $x_n^{(i)}$ move away from the original orbits for the uncoupled maps. In the other aspect the values of the factors $(1-\epsilon)$ and $\epsilon/2$ change also. It is clear that the latter could alter the Λ only in the magnitude order $O(\epsilon)$, and the former must be responsible for the variation of Λ. Then the matrix $A^{(N)}$ could be considered to be diagonal with the elements

$$A_{ij}^{(N)} \simeq \delta_{ij} \prod_{n=0}^{N-1} f'(x_n^{(i)}) \,.$$

Letting $N = T$ with T the period of the motion, the Lyapunov exponents of total number of L are approximated as

$$\lambda^{(i)} \simeq \frac{1}{T} \sum_{n=0}^{T-1} \ln |f'(x_n^{(i)})| \qquad i = 1, 2, \ldots, L \tag{10}$$

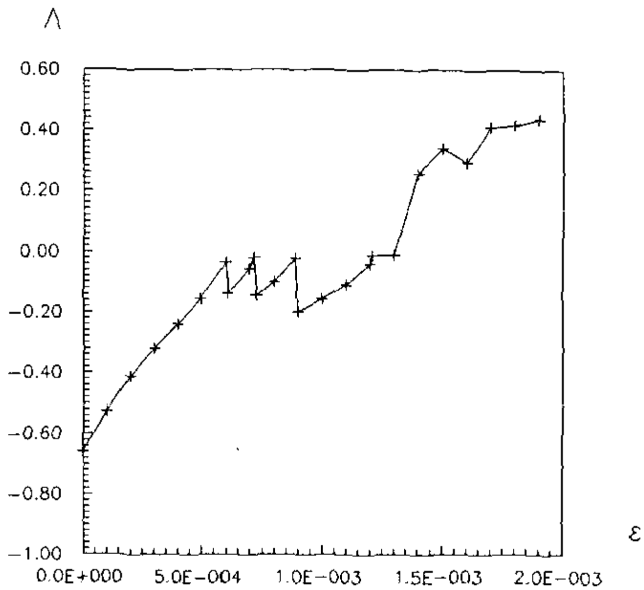

Figure 1. The curve of the first Lyapunov exponent Λ for the map lattice system, obtained by the standard technique, versus the coupling strength ϵ. Here $L = 173$, and $\mu = 3.831$.

which is coincident with the formula for the single map (3). As long as the orbits $x_n^{(i)}$ for each map in the lattice are found, first Lyapunov exponents can be calculated by the formula (10).

3. Mode-analytic method

In this section we denote the period-3 orbit of the single map (3) as $x_0 = f(x_2)$, $x_1 = f(x_0)$, and $x_2 = f(x_1)$. Letting $\epsilon = 0$, we have an uncoupled map lattice. For any random initial conditions each individual map in this lattice would settle down to this period orbit after a short transient time. That means that the iteration of each lattice point would visit x_0, x_1, and x_2 successively. For a lattice of large size, however, the 'phases' of all lattice points could hardly be coherent. In other words, the state of the uncoupled map lattice is mostly inhomogeneous. For a given time each lattice point must visit one of the three values x_0, x_1, or x_2, being denoted as 0, 1 and 2, respectively. The state of the lattice at a certain time could then be described by a sequence of total number L, for instance

$$0,\ 0,\ 1,\ 2,\ 1,\ 0,\ 2,\ 2,\ 0,\ 2,\ \ldots,\ 1. \tag{11}$$

As the couplings between the neighbours switch on, the orbit of each local map might change. However, as long as the coupling strength is weak enough the modification of the orbits must be very small, so we can still denote the state of the lattice as the sequence (11). In order to calculate the modified orbit for a given lattice point, we may assume that its two neighbours stick to their original orbits. Hence we can consider the following simple set of three maps:

$$\begin{cases} x_{n+1}^{(i-1)} = f(x_n^{(i-1)}) \\ x_{n+1}^{(i)} = (1-\epsilon)f(x_n^{(i)}) + \frac{\epsilon}{2}[f(x_n^{(i-1)}) + f(x_n^{(i+1)})] \\ x_{n+1}^{(i+1)} = f(x_n^{(i+1)}) \,. \end{cases} \tag{12}$$

This set of maps can be studied numerically as well as analytically, and the stable period orbits, which certainly depend on the initial condition, can be obtained. The initial configuration is denoted as (k,l,m), with k, l and m taken to be 0, 1 or 2, which means that the initial values for the lattice points $(i-1)$, i and $(i+1)$ are set to x_k, x_l and x_m, respectively. For a given (k,l,m) the three-dimensional maps are called a mode. It is clear that within three iterations each point of the lattice will visit the three values x_0, x_1 and x_2, though their starting values are different. Without losing generalities we may assume that $l = 0$ and $m \geqslant k$. Then for the period-3 window there are six modes altogether, namely

$$(0,0,0),\ (0,0,1),\ (0,0,2),\ (1,0,1),\ (1,0,2),\ (2,0,2)\,. \tag{13}$$

In the case of the sequence (11), for example, the map at the second point of the lattice, $i = 2$, is located at the mode $(0,0,1)$. As to the map at point $i = 3$ we have the mode $(0,1,2)$ which becomes $(2,0,1)$ after two iterations, and could be rewritten as $(1,0,2)$, appearing in (13). For a given mode (12) the motion of the middle map can be described by

$$x_{n+1}^{(i)} = F(x_n^{(i)}) \tag{14}$$

where

$$F(x_n^{(i)}) = (1-\epsilon)f(x_n^{(i)}) + \epsilon t_n^{(i)} \tag{15}$$

and

$$t_n^{(i)} = \tfrac{1}{2}[f(x_n^{(i-1)}) + f(x_n^{(i+1)})] \tag{16}$$

while the motions of its neighnours, the lattice points $(i-1)$ and $(i+1)$, are in the stable period-3 orbit since each of them is essentially an independent single map (3). The form of $F(x)$ must return to itself after every three time steps because $t_n(i)$ changes its value with period 3. Hence the function

$$X_{s+1} = F^{(3)}(X_s) \equiv F\{F[F(X_s)]\} \tag{17}$$

defines a one-dimensional map in the interval $X_s \in [0,1]$ with two parameters μ and ϵ. The mode (k,l,m) is called stable if there is a stable fixed point near $X_s = x_l$, otherwise it is unstable. In fact, as we will see later, there is a well defined value of $\epsilon_t = \epsilon_t(\mu)$ at which the map (17) undergoes a tangent bifurcation. The exact meaning of the statement 'a coupling strength being weak enough' in this paper is that $\epsilon < \epsilon_t$. In this case a fixed point near x_l is stable, and the mode (k,l,m) then must be stable too. For a stable mode (k,l,m) the maps (12) must return to this mode after any number of periods.

Let us determine the tangent bifurcation point for each mode in (13). In the case of $\epsilon = 0$ we may consider only the single map (3) for which the leftmost point the μ_0 of the period-3 window in the μ-axis corresponds to the tangent bifurcation

$$\begin{cases} f\{f[f(x)]\} = x \\ \dfrac{\mathrm{d}}{\mathrm{d}x} f\{f[f(x)]\} = 1 \,. \end{cases} \tag{18}$$

When $\mu > \mu_0$ the tangent bifurcation point takes place at some value of $\epsilon_t > 0$. This could be calculated by considering

$$\begin{cases} F\{F[F(x)]\} = x \\ \dfrac{\mathrm{d}}{\mathrm{d}x} F\{F[F(x)]\} = 1 \end{cases} \tag{19}$$

which are equations characterizing the dependence of ϵ_t on μ at the tangent bifurcation point. Letting

$$y(x,\mu,\epsilon) = F\{F[F(x)]\} - x \tag{20}$$

we have from (18) that

$$\begin{cases} y(x_0,\mu_0,0) = 0 \\ \dfrac{\partial y(x_0,\mu_0,0)}{\partial x} = 0 \,. \end{cases} \tag{21}$$

Hence we obtain from (19) by the implicit function theorem that

$$\frac{\partial y(x_0,\mu_0,0)}{\partial \mu}\mathrm{d}\mu + \frac{\partial y(x_0,\mu_0,0)}{\partial \epsilon}\mathrm{d}\epsilon = 0 \,. \tag{22}$$

A simple calculation gives

$$\frac{\partial y(x_0,\mu_0,0)}{\partial \mu} = \frac{1}{\mu_0}\left\{x_0 + \left[x_2 + x_1\frac{\mathrm{d}}{\mathrm{d}x}f(x_1)\right]\frac{\mathrm{d}}{\mathrm{d}x}f(x_2)\right\} \tag{23}$$

and

$$\frac{\partial y(x_0,\mu_0,0)}{\partial \epsilon} = t_2 - x_0 + \left[t_1 - x_2 + (t_0 - x_1)\frac{\mathrm{d}}{\mathrm{d}x}f(x_1)\right]\frac{\mathrm{d}}{\mathrm{d}x}f(x_2) \,. \tag{24}$$

For a given mode $(k, 0, m)$ we have

$$t_n = \tfrac{1}{2}(x_K + x_M) \tag{25}$$

where

$$K = (k+n+1) \mod 3 \qquad M = (m+n+1) \mod 3 \,. \tag{26}$$

For the period-3 window $\mu_0 = 1 + \sqrt{8}$ and $x_0 \simeq 0.1599$, $x_1 \simeq 0.5144$, $x_2 \simeq 0.9563$, we can obtain from (23)–(25)

$$\frac{\mathrm{d}\epsilon}{\mathrm{d}\mu} = \begin{cases} \infty & \text{for} \quad (0,0,0) \\ 0.7071 & \text{for} \quad (0,0,2) \\ 0.4714 & \text{for} \quad (0,0,1) \\ 0.3536 & \text{for} \quad (2,0,2) \\ 0.2828 & \text{for} \quad (1,0,2) \\ 0.2357 & \text{for} \quad (1,0,1)\,. \end{cases} \tag{27}$$

The tangent bifurcation takes place at

$$\epsilon_t \simeq \frac{\mathrm{d}\epsilon}{\mathrm{d}\mu}(\mu - \mu_0)\,. \tag{28}$$

If $\epsilon > 0.7071(\mu - \mu_0)$, only the mode $(0,0,0)$ is stable, and the stable state for the coupled map lattice (1) is the coherent period state. As we mentioned above the map lattice system in this case will in fact display a chaotic motion because of the extremely long relaxation time. If $0.4714(\mu - \mu_0) < \epsilon < 0.7071(\mu - \mu_0)$, we have two stable modes $(0,0,0)$ and $(0,0,2)$, and it seems to us that the stable state of (1) may not be a coherent one. However, a sequence of form (11) can never be constructed by these two modes, and a third mode has to be added. So the motion of the system (1) in this case is still chaotic. When $\epsilon < 0.4714(\mu - \mu_0)$ we have at least three stable modes $(0,0,0)$, $(0,0,2)$, and $(0,0,1)$, and an inhomogeneous state for (1) could be constructed easily, such as

$$0,\ 0,\ 1,\ 1,\ 1,\ 2,\ 2,\ 2,\ 0,\ 0,\ \ldots,\ 0\,. \tag{29}$$

Hence we find that the boundary of the period-3 window for (1) can be defined as

$$\epsilon_w \simeq 0.4714(\mu - \mu_0)\,. \tag{30}$$

For $\mu = 3.831$ we have $\mu - \mu_0 \simeq 0.00273$, and the depth of the window is $\epsilon_w \simeq 1.21 \times 10^{-3}$, coinciding with the numerical results in figure 1. According to (28) the other three peaks of the curve in figure 1 should be $\epsilon_1 \simeq 0.2357 \times 0.00273 \simeq 6.43 \times 10^{-4}$, $\epsilon_2 \simeq 7.72 \times 10^{-4}$, and $\epsilon_3 \simeq 9.65 \times 10^{-4}$, respectively.

For a given set of the initial conditions each lattice point with its two neighbours must very quickly reach one of the modes. As long as the size of the lattice system is large enough every mode could appear. If a mode is unstable for the given μ and ϵ it will collapse, and eventually relax to a stable mode. For the period-3 window three or more stable modes can always be organized to a long chain. Then the first Lyapunov exponent for the map lattice (1) must be the largest Lyapunov exponents within the stable modes.

The numerical work to calculate the first Lyapunov exponent has been done using the mode-analytic method proposed above. The results are shown in figure 2. It is easy to find that in the interval $\epsilon \in [0, \epsilon_w]$ they coincide with that of figure 1, the results obtained by the standard technique. When $\epsilon > \epsilon_w$ the standard method gives positive first Lyapunov exponent while the mode-analytical method gives the coherent orbit's results.

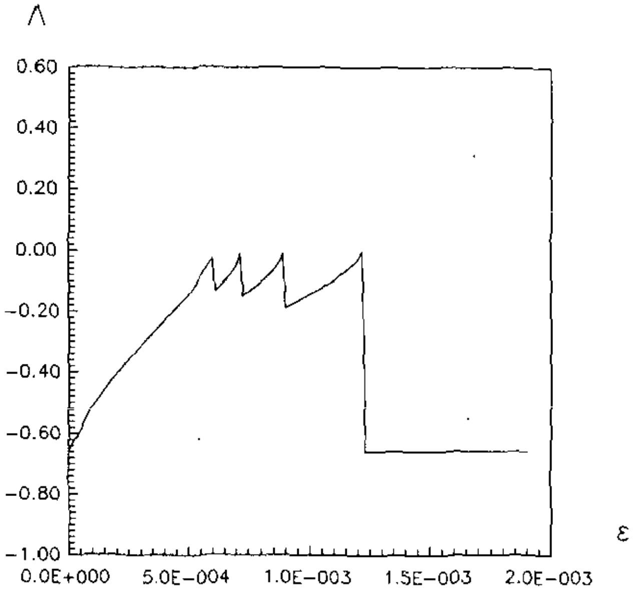

Figure 2. *The first Lyapunov exponent, calculated by the mode-analytic method, proposed in the present paper.*

4. Discussions

(i) The value of $\mu = 3.831$ is less than μ_c, corresponding to the superstable orbit. For $\mu > \mu_c$ the criterion is still correct. The numerical results for $\mu = 3.833 > \mu_c$, obtained both by the standard technique and by the mode-analytical method, are shown in figure 3. It is evident that they coincide each other perfectly in the interval $\epsilon < \epsilon_w$.

(ii) By numerical simulations for map lattice (1) we find the depth of the period-3 window. The results, shown in figure 4, are in agreement with (30). Further increasing the value of μ we may arrive the period-doubling bifurcation point μ_b. For $\mu > \mu_b$ the period of the single map (3) becomes 6, and the mode-analytic method proposed here is no longer valid.

(iii) The method is applicable to the leading period of every period window, but the formulae (23) and (24) should be generalized respectively to

$$\frac{\partial y(x_0, \mu_0, 0)}{\partial \mu} = \frac{1}{\mu_0} \sum_{i=0}^{T-1} x_i \prod_{j=1}^{T-i} \frac{\mathrm{d}}{\mathrm{d}x} f(x_j) \tag{31}$$

and

$$\frac{\partial y(x_0, \mu_0, 0)}{\partial \epsilon} = \sum_{i=0}^{T-1} (t_{i-1} - x_i) \prod_{j=1}^{T-i} \frac{\mathrm{d}}{\mathrm{d}x} f(x_j) \tag{32}$$

where the ts are defined in (25) and (26) with period T instead of 3.

(iv) The discussion in this paper is confined to the stable periodic state of the coupled map lattice. The approach, however, might also be helpful to the discussion of the window–chaos transition, studied in [7].

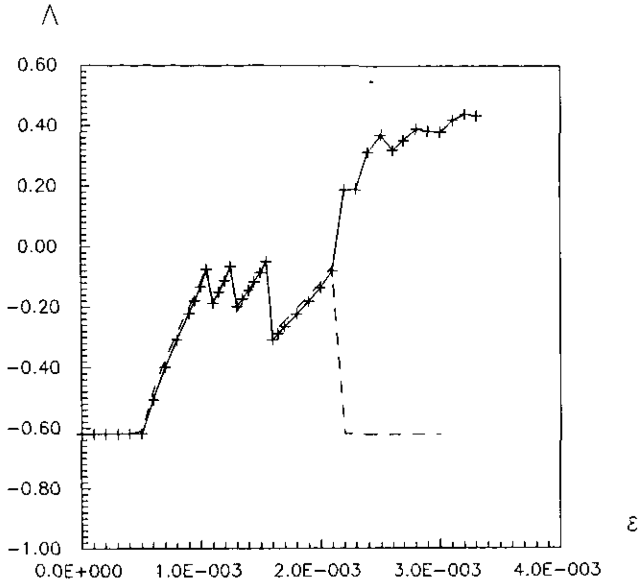

Figure 3. For $\mu = 3.834$ the numerical calculation for the first Lyapunov exponent Λ. The full curve stands for the standard method and the broken curve for the mode-analytic method.

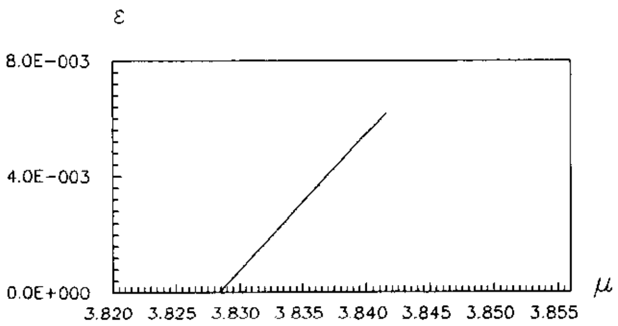

Figure 4. The depth of the period-3 window for the coupled map lattice.

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